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On the rippling of small waves: a harmonic nonlinear nearly resonant interaction

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We show that the rippling often observed on small progressive gravity waves can be explained in terms of a nearly resonant harmonic nonlinear interaction. The resonance condition is that the phase speeds of the two waves must be nearly identical. The inviscid analysis is generalized to any order in a small parameter proportional to the wave steepness. Wave tank measurements provide experimental evidence for most of the predicted results. The phenomenon of resonant rippling is further shown to be not just peculiar to capillary-gravity waves, but in fact possible for any weakly nonlinear dispersive wave system whose dispersion relation has discrete pairs of solutions nearly satisfying the resonance conditions.

1. Introduction

The often observed appearance of capillary waves on the forward face of a steep gravity wave has been the subject of several recent analyses. Longuet-Higgins (1963) considered the problem in which, when a progressive gravity wave approaches its maximum steepness and develops a sharpened crest, the surface tension must be at least locally important. This results in a travelling pressure disturbance which gives rise to a train of capillary waves ahead of the crest, or on the forward face. The properties of these wavelets were then determined by considering them to be a small perturbation on some basic flow, which was taken to be that due to the gravity wave itself. More recently, Crapper (1970) reconsidered the phenomenon by considering the capillary waves as stationary waves on a slowly varying (spatially) running stream, but used as the perturbation his *exact* nonlinear capillary wave solution (Crapper 1957). His analysis used the more recent method due to Whitham (1965*a*, *b*) involving the use of an 'averaged Lagrangian'. Both Longuet-Higgins and Crapper appealed to some earlier experiments of Cox (1958) for qualitative, if not quantitative, agreement with their respective theories.

The success and applicability of these similar theories requires the acceptance of two suppositions: that the ratio of the wavelength of the disturbance to that of the basic flow gravity wave be very small, and second, that the capillary wave phase speed be identical to that of the gravity wave in order that the motion may be considered steady in a uniformly translating co-ordinate system. Under these conditions (in particular, the former) their analyses doubtless are accurate, and we can have no quarrel with the results.

On the other hand, both authors agree that it is at best marginal to use the observations of Cox for verification. For the frequencies of the waves in his experiments ($\sim$ 6·6 c/s), there is an appreciable effect of surface tension on the 'gravity wave' and further, the ratio of wavelengths is not as small as one would ideally want. It is desirable to investigate this situation for a more appropriate gravity wave (longer), and accordingly a laboratory study was begun. Unfortunately, any results relevant to the original intent of the investigation were quickly seen to be nearly impossible to obtain because of the limited size of the experimental facilities and other problems. The length of the wave tank forced me to use fairly short 'gravity' waves which in fact *are* influenced more or less by surface tension.

Some preliminary measurements involving the rippling on waves with fundamental frequencies in the range of 5–7 c/s provided some rather surprising facts, the details of which will be apparent in subsequent sections of this paper. Briefly, if a wave maker is driven with purely sinusoidal motion at some fairly small amplitude, progressive waves having the wavelength appropriate to that driving frequency appear in the immediate vicinity of the source as fairly clean nearly sinusoidal waves. As they progress away from the source, ripples begin to form, and can be seen (by eye) to grow in amplitude as they progress. It is not too difficult (though tedious) with proper electronics and technique to measure the phase speed of the fundamental and (by judicious filtering) the phase speed of the ripples. The speeds are almost always different! The wave train is not spatially steady. At any fixed distance from the source, however, the wave form is *temporally* steady, and a measurement of the amplitude spectrum of the wave form seen at a fixed point was observed to contain only harmonics of the wave maker frequency. Finally, the ripples are not confined to the forward face of the longer wave, but in fact (see figure 1, plate 1) are ubiquitous.

That these observations are not at all consistent with the above-mentioned theories is of no consequence. They should not be. Indeed, what is needed for their explanation is a theory in which those assumptions of steadiness (spatially) and wavelength scale separation are abandoned. What we wish to investigate here, then, is the spatial modulation, or wrinkling, of a single Fourier progressive wave mode by a nonlinear harmonic interaction.

It will become apparent in the following parts of this paper that the phenomenon of rippling is not just a peculiarity of capillary-gravity waves, but much more general.† It will occur for any weakly nonlinear dispersive wave system for which the dispersion relation is such that the wavenumber is a double-valued function of the phase speed, or for which at discrete frequencies) if the dispersion relation is $\omega = f(k)$, then $n\omega = f(nk)$ for some finite range of integers n, that is, the medium must admit of free waves and their free nth harmonic that can travel at identical phase speeds. That this is true for capillary-gravity waves was pointed

† Drazin (1970) investigated some properties of nonlinear Kelvin–Helmholtz interfacial waves. That harmonic resonances of the kind described in the present paper are possible in his problem is clear from the form of his dispersion relation, as well as from his paragraph on page 326, following equation (38), which of course is the condition for second harmonic resonance.

out by Wilton (1915), and subsequent analysis and experiment have been performed for $n = 2$ (second harmonic resonance) by McGoldrick (1970*a*, *b*).

It is well known that the dynamical equations governing the propagation of capillary-gravity waves are weakly nonlinear, and the algebraic details of carrying out their expansion by perturbation methods to sufficient accuracy anticipated here are prohibitively difficult. For this reason, we shall present in §2 a model equation having nearly identical properties: weakly nonlinear, and having a simple dispersion relation admitting of discrete harmonic resonances. We shall then in the next section solve the model equation (subject to initial conditions appropriate to the anticipated experiment) using a straightforward perturbation scheme with multiple space scales. The results of the analysis will show clearly that the rippling may be interpreted as a nonlinear resonant interaction between a fundamental wave mode and one of its temporal (but not necessarily spatial) harmonics.

It is not our intention to solve the water wave equations. We wish only to use some of the observed properties of these interactions for partial verification of the solutions of the analogous model equation. Accordingly, in §4, we turn to the wave tank and present some experimental measurements of ripple formation that confirm qualitatively some of the predictions of the theoretical development of §3.

Kim & Hanratty (1971) have presented some experimental results that are similar to some of those that we shall present here. They have shown that an initially sinusoidal wave propagating in shallow water (0·65 cm deep) develops seven additional crests by the time it has progressed about 31 cm from its wave maker source, which indicates a growth of an eighth harmonic component with distance. They also show that at different (but still small) depths, it is possible to create third and fourth harmonic distortion. They interpret the latter resonances as the result of a quadratic interaction rather than a cubic or quartic interaction like those to be described in this paper. We shall return to this point in §4.

Finally, in the concluding section, we shall, among further comments, attempt to place the theory and experiment of this paper in proper perspective to the theories of Longuet-Higgins and Crapper.

2. Preliminary theoretical considerations

The equations governing the propagation of surface waves under the combined influence of gravity and surface tension are well known. If a prime is used to denote dimensional variables, then everything may be made dimensionless according to the scheme $\zeta = \zeta'/a$, $\phi = \phi'/ac$, $\mathbf{x} = \kappa\mathbf{x}'$, $t = \Omega t'$, $k = k'/\kappa$, $\omega = \omega'/\Omega$, where a, c, k and Ω are typical dimensional amplitude, phase speed, wavenumber and frequency. With ζ' the departure of the free-surface elevation from equilibrium and ϕ' the velocity potential for the motion, the exact dimensionless kinematical boundary condition at the free surface becomes

$$\zeta_t - \phi_z + \epsilon\nabla\phi\,.\,\nabla\zeta = 0 \quad \text{at} \quad z = \epsilon\zeta, \tag{2.1}$$

where $\epsilon = a\kappa$. The dynamical boundary condition is obtained from the substantial derivative of the Bernoulli equation evaluated at the free surface, which is exactly

$$\phi_{tt}+\lambda\phi_z-\mu DF(\zeta)/Dt+\epsilon(\nabla\phi.\nabla\phi)_t+\tfrac{1}{2}\epsilon^2[\nabla\phi.\nabla(\nabla\phi.\nabla\phi)] = 0 \quad \text{at} \quad z = \epsilon\zeta. \tag{2.2}$$

In (2.2), $\lambda = g\kappa/\Omega^2$, $\mu = \gamma\kappa^3/\Omega^2$, $D/Dt = \partial/\partial t+\epsilon\nabla\phi.\nabla$, γ is the surface tension coefficient divided by the density, $F(\zeta)$ is the sum of the principal curvatures of the surface, and the atmospheric pressure is as usual taken to be constant. Equations (2.1) and (2.2) together with Laplace's equation and the condition that $\nabla\phi$ vanishes as $z \to -\infty$ govern the problem exactly.

It is usual in problems of this sort to expand the dependent variables in series in the parameter ϵ such as

$$\left.\begin{aligned} \zeta &= \zeta^{(1)}+\epsilon\zeta^{(2)}+\epsilon^2\zeta^{(3)}+\ldots, \\ \phi &= \phi^{(1)}+\epsilon\phi^{(2)}+\epsilon^2\phi^{(3)}+\ldots, \end{aligned}\right\} \tag{2.3}$$

where $\zeta^{(n)}$ and $\phi^{(n)}$ are $O(1)$ quantities. Without performing any of the details, two essential points become obvious. First, the problem of determining ϕ is weakly nonlinear. That is, quadratic, cubic, ... terms in ϕ always appear with coefficients $\epsilon, \epsilon^2, \ldots$, respectively, as is well known. Further, the linearized problem ($\epsilon \to 0$) becomes, to $O(1)$,

$$\left.\begin{aligned} &\zeta^{(1)}_t-\phi^{(1)}_z = 0 \quad \text{at} \quad z = 0, \\ &\phi^{(1)}_{tt}+\lambda\phi^{(1)}_z-\mu\phi^{(1)}_{xxz} = 0 \quad \text{at} \quad z = 0, \\ &\phi^{(1)}_{xx}+\phi^{(1)}_{zz} = 0, \\ &\{\phi^{(1)}_x, \phi^{(1)}_z\} \to 0 \quad \text{as} \quad z \to -\infty, \end{aligned}\right\} \tag{2.4}$$

where we have chosen x as the direction of propagation of a wave of the form $\zeta^{(1)} = \mathrm{Re}\{A\, e^{i(kx-\omega t)}\}$. Equations (2.4) then easily yield the well-known dispersion relation which, in dimensional terms, is

$$\omega' = f(k') = (gk'+\gamma k'^3)^{\frac{1}{2}}, \tag{2.5}$$

and the phase speed of the wave is given by

$$c' = f(k')/k' = (g/k'+\gamma k')^{\frac{1}{2}}. \tag{2.6}$$

The second essential point is that for any phase speed greater than the minimum value allowed by (2.6) (i.e. $c'_m = (4g\gamma)^{\frac{1}{4}}$) the dispersion relation (2.6) allows two wavenumbers as solutions. In particular, there exists a sequence of wavenumbers for which harmonically related waves have identical phase speeds. A fundamental free wave with wavenumbers $k' = (g/n\gamma)^{\frac{1}{2}}$ and its nth harmonic free wave with wavenumber $nk' = (ng/\gamma)^{\frac{1}{2}}$ propagate at identical phase speeds,

$$c' = (ng\gamma)^{\frac{1}{4}}\,(1+1/n)^{\frac{1}{2}} \quad \text{for} \quad n = 2, 3, 4, \ldots.$$

That this situation admits of singularities in the nonlinear problem has been pointed out by Wilton (1915). The interpretation of the $n = 2$ singularity, which arises from the quadratically nonlinear terms (at $O(\epsilon)$), has been shown by McGoldrick (1970*a*, *b*) to be a special case of resonant interaction. It is clear that the higher order singularities can also be interpreted as nth harmonic resonances, and for the resolution of the problem, the usual perturbation analysis must be carried to $O(\epsilon^{n-1})$. For capillary-gravity waves, at least, the analysis becomes progressively (and prohibitively) more tedious at successive orders. Admitting to a strong aversion to unnecessarily complicated algebraic manipulations, we shall present with no further apology a simple model possessing both of the essential features and investigate in detail not just the resonant solutions, but, more to the point of the experimental observations to be presented below, the properties of the interaction in the neighbourhood of resonance will be clarified.

Let us consider, then, the following one-dimensional model which admits of dispersive plane wave solutions:

$$u_{tt}-u_{xx}+u+\tfrac{1}{4}u_{xxxx} = 3\epsilon u^2+\beta\epsilon^2u^3+\ldots, \tag{2.7}$$

which may be simplified for later reference by writing

$$\mathscr{L}\{u\} = \epsilon\mathscr{N}\{u,\epsilon\},$$

where $\mathscr{L}$ is a linear differential operator in x and t, $\mathscr{N}$ is a nonlinear operator, and $0 < \epsilon \ll 1$ is an arbitrary small parameter. The constants in (2.7) have been chosen for subsequent algebraical simplicity. If we now write $u = u^{(1)}+\epsilon u^{(2)}+\epsilon u^{(3)}+\ldots$, where $u^{(n)} = O(1)$, then substitution into (2.7) yields the following sequence of problems:

$$\left.\begin{aligned} O(1)&: \quad \mathscr{L}\{u^{(1)}\} = 0,\\ O(\epsilon)&: \quad \mathscr{L}\{u^{(2)}\} = 3u^{(1)2},\\ O(\epsilon^2)&: \quad \mathscr{L}\{u^{(3)}\} = 6u^{(1)}u^{(2)}+\beta u^{(1)3},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{2.8}$$

which is as usual solved successively. Choosing $u^{(1)} = A\,e^{i\theta}+A^*e^{-i\theta}$ for the $O(1)$ solution, with A a constant amplitude and $\theta = kx-\omega t$ the phase, then the $O(1)$ equations yield solely the simple dispersion relation

$$\omega = f(k) = 1+\tfrac{1}{2}k^2. \tag{2.9}$$

This allows harmonically related free waves to travel at the same phase speed for the sequence of discrete frequencies and corresponding wavenumbers which are solutions of $n\omega = f(nk)$, namely

$$\omega_n = 1+1/n, \quad k_n = (2/n)^{\frac{1}{2}}. \tag{2.10}$$

Using the solution for $u^{(1)}$, then the particular solution of the $O(\epsilon)$ problem of (2.8) is

$$u^{(2)} = \frac{3}{f^2(2k)-(2\omega)^2}\{A^2e^{2i\theta}+A^{*2}e^{-2i\theta}\}+6AA^*, \tag{2.11}$$

which is bounded unless (2.10) with $n = 2$ is satisfied. This of course is the familiar second harmonic resonance in which a fundamental and its second harmonic are both free waves.

To be more precise, consider the coefficient of the forced oscillations in (2.11). Suppose the wavenumber (whence the frequency) is close to resonance for $n = 2$, and with $\omega_2 = \frac{3}{2}$, $k_2 = 1$, write $k = k_2(1+\delta)$, where δ is a small number. Then to lowest order in δ, the solution for u correct to $O(\epsilon)$ is

$$\begin{aligned} u &= u^{(1)}+\epsilon u^{(2)}\\ &= A\,e^{i\theta}+A^*e^{-i\theta}+6\epsilon AA^*-\frac{\epsilon}{4\delta}\{A^2e^{2i\theta}+A^{*2}e^{-2i\theta}\}. \end{aligned} \tag{2.12}$$

If the amount of 'detuning from resonance', characterized by δ, is such that $\delta = O(\epsilon)$, then the forced harmonic solutions are $O(1)$, contrary to the spirit of the original expansion. This suggests that, near resonance, the second harmonic terms should be considered as part of the $O(1)$ term in the expansion for u. Some care must be exercised on this point, because the terms in curly brackets of (2.12) are *not* free waves for $\delta \neq 0$; they are not exact solutions of the $O(1)$ equations. They are, however, arbitrarily close to free solutions, and proper allowance for this small discrepancy will be explained in the next section.

If the frequency is *not* close to second harmonic resonance, and $u^{(2)}$ given by (2.11) is $O(1)$, then the $O(\epsilon^2)$ problem of (2.8) can be solved, with particular integral containing, among other things, a forced third harmonic component, or (with M_0 an $O(1)$ constant)

$$u^{(3)} = \frac{M_0}{f^2(3k)-(3\omega)^2}\{A^3 e^{3i\theta} + A^{*3} e^{-3i\theta}\} + \ldots, \tag{2.13}$$

which is bounded unless the frequency is sufficiently close to the third harmonic resonant frequency $\omega_3 = \frac{4}{3}$, with $k_3 = (\frac{2}{3})^{\frac{1}{2}}$. If the actual wavenumber is $k = k_3(1+\delta)$, δ again small, then using (2.13) in the expansion for u to $O(\epsilon^2)$, we have

$$u = A e^{i\theta} + A^* e^{-i\theta} + \frac{3\epsilon}{f^2(2k)-(2\omega)^2}\{A^2 e^{2i\theta} + A^{*2} e^{-2i\theta}\} + 6\epsilon A A^* + \frac{M_1 \epsilon^2}{\delta}\{A^3 e^{3i\theta} + A^{*3} e^{-3i\theta}\} + O(\epsilon^2) \tag{2.14}$$

with M_1 another constant. Here again, if the detuning δ is of order ϵ^2 then the forced third harmonic terms are $O(1)$, and should be absorbed in the $O(1)$ solution $u^{(1)}$, again remembering that the forced third harmonics are not quite, but sufficiently close to, free waves.

The pattern that emerges is clear, and can easily be generalized. If the actual frequency and wavenumber differ from an nth harmonic pair given by (2.10) according to

$$|\omega - \omega_n| = O(\epsilon^{n-1}), \quad |k-k_n| = O(\epsilon^{n-1}) \quad (n = 2, 3, \ldots), \tag{2.15}$$

then the nth harmonic forced wave will be $O(1)$, and should be absorbed in the $O(1)$ term in the expansion, $u^{(1)}$. As we shall see below, under these conditions, the precise nature of the $O(1)$ solution will not emerge until the sequence of problems (2.8) is completed to $O(\epsilon^{n-1})$.

3. The analysis near resonance

3.1. *Second harmonic near-resonance*

In order to resolve the difficulties arising near resonance from the straightforward perturbation expansion of the preceding section, we shall make liberal use of the method of multiple scales. Formally, assume that the complex amplitudes of the waves depend on a sequence of slow time scales given by

$$T_1 = \epsilon t, \quad T_2 = \epsilon^2 t, \ldots, T_n = \epsilon^n t,$$

and a sequence of long space scales $X_1 = \epsilon x$, $X_2 = \epsilon^2 x, \ldots$. Then time derivatives

$\partial/\partial t$ are replaced by the sequence of derivatives $\partial/\partial t+\epsilon\partial/\partial T_1+\epsilon^2\partial/\partial T_2+\ldots$ and spatial derivatives $\partial/\partial x$ become $\partial/\partial x+\epsilon\partial/\partial x_1+\epsilon^2\partial/\partial X_2+\ldots$. Then the linear operator $\mathscr{L}$ of (2.7) becomes upon expansion

$$\begin{aligned}\mathscr{L}\{u\} \to\ & u_{tt}-u_{xx}+u+\tfrac{1}{4}u_{xxxx}\\ &+\epsilon\{2u_{tT_1}-2u_{xX_1}+u_{xxxX_1}\}\\ &+\epsilon^2\{(2u_{tT_2}-2u_{xX_2}+u_{xxxX_2})\\ &+(u_{T_1T_1}-u_{X_1X_1}+\tfrac{3}{2}u_{xxX_1X_1})\}+\ldots.\end{aligned} \tag{3.1}$$

Then with $u = u^{(1)}+\epsilon u^{(2)}+\epsilon^2 u^{(3)}+\ldots$, the following sequence of problems arises to replace (2.8):

$$\left.\begin{aligned} O(1)&: \quad \mathscr{L}\{u^{(1)}\} = 0,\\ O(\epsilon)&: \quad \mathscr{L}\{u^{(2)}\} = 3u^{(1)2}-(2u^{(1)}_{tT_1}-2u^{(1)}_{xX_1}+u^{(1)}_{xxxX_1}),\\ O(\epsilon^2)&: \quad \mathscr{L}\{u^{(3)}\} = 6u^{(1)}u^{(2)}+\beta u^{(1)3}\\ &\qquad -(2u^{(2)}_{tT_1}-2u^{(2)}_{xX_1}+u^{(2)}_{xxxX_1})\\ &\qquad -(2u^{(1)}_{tT_2}-2u^{(1)}_{xX_2}+u^{(1)}_{xxxX_2})\\ &\qquad -(u^{(1)}_{T_1T_1}-u^{(1)}_{X_1X_1}+\tfrac{3}{2}u^{(1)}_{xxX_1X_1}),\\ &\qquad \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \tag{3.2}$$

The linear problem is unaffected by this expansion, and if we choose

$$u^{(1)} = A(T_1,\ldots,X_1,\ldots)\,e^{i\theta}+[*],$$

the sole result of this problem is the dispersion relation (2.9). To this order, the slow time and long space scale nature of A is indeterminate.

If the frequency is not near a solution of the resonance condition $2\omega = f(2k)$, then the particular integral of the $O(\epsilon)$ equation is (2.11) again, but in addition, the terms in parentheses, which must sum to zero in order that $u^{(2)}$ be bounded, yield

$$A_{T_1}+UA_{X_1} = 0, \tag{3.3}$$

with general solution $A(X_1,T_1) = \mathscr{F}(X_1-UT_1)$. U is the group velocity

$$\partial\omega/\partial k = k.$$

That is, to this order, in a frame of reference translating with the group velocity the local amplitude is a constant.

If the frequency is near to that of second-harmonic resonance, the results of the last section suggest that the $O(1)$ solution be modified to allow the possibility of an $O(1)$ near-second harmonic constituent. The particular form of the modification is not immediately apparent, however, and we must turn to preliminary experimental observations for some enlightenment. If in a wave tank a wave maker is made to oscillate with constant amplitude with a single frequency near the resonant frequency, then after the initial disturbance† has propagated

† The problem of the initial disturbance is of course unsteady, and the time derivatives of any order are not negligible. Inclusion of these derivatives at any appropriate order in the problem will easily yield amplitude equations similar to those we shall produce below, (3.10), (3.25), (3.41), (3.43), etc., but in which the left-hand sides must be replaced by $A_{T_n}+U_A A_{X_n}$ and $B_{T_n}+U_B B_{X_n}$, respectively. It is easy to generate these equations but difficult to integrate them from whatever initial data is prescribed, say A and B for all T_n at a wave maker, $X_n = 0$. We are still working on this problem and shall return to it at a later date.

sufficiently far, the wave form in the vicinity of the wave maker is seen to be steady in time, but variable or modulated in space. If a wave-measuring probe is placed in a fixed position, then the amplitude of the waves is not observed to change with time, and Fourier analysis of the steady wave form with sharp electronic filters indicates that the amplitude spectrum consists solely of the fundamental (wave maker) frequency and its discrete (but exact) temporal harmonics. Since these conditions obtain in all of the experiments to be reported in the next section, we shall make the assumption for the remainder of this paper that the amplitudes are not functions of any of the slow time scales, or $\partial A/\partial T_n = 0$, and investigate the details of the steady spatially modulated solutions of (3.2).

For second harmonic near-resonance, then, we choose the $O(1)$ wave field as

$$u^{(1)} = A(X_1)\, e^{i\theta_1} + \tfrac{1}{2}B(X_1)\, e^{i\theta_2} + [*], \tag{3.4}$$

with phase functions $\theta_1 = k(\omega)x - \omega t$ and $\theta_2 = k(2\omega)x - 2\omega t$. The wavenumbers $k(\omega)$ and $k(2\omega)$ must be determined from the inversion of the dispersion relation (2.9) in order that (3.4) satisfies the $O(1)$ problem of (3.2) exactly, ensuring that A and B represent free waves. For frequencies ω close to $\omega_2(=\frac{3}{2})$, then, writing $\omega = \omega_2(1+\delta)$ with δ a small number, it is a simple matter to show that to the lowest order in δ the wavenumbers in (3.4) must be

$$k(\omega) = k_2(1 + c_2\delta/U_A), \quad k(2\omega) = 2k_2(1+c_2\delta/U_B), \tag{3.5}$$

where c_2 is the phase speed ω_2/k_2 evaluated at resonance, U_A is the group speed $d\omega/dk|_{\omega_2}$ evaluated at the resonant frequency, and U_B is the group speed $d\omega/dk|_{2\omega_2}$ evaluated at *twice* the resonant frequency. Note that even though the frequencies are harmonic, the wavenumbers are *not*, but in fact

$$\frac{k(2\omega)}{k(\omega)} = 2\left\{1 + c_2\delta\left(\frac{1}{U_B} - \frac{1}{U_A}\right)\right\}, \tag{3.6}$$

which is harmonic only at exact resonance, $\delta = 0$.

Substituting the free wave solution (3.4) into the $O(\epsilon)$ equations of (3.2), we get

$$\begin{aligned}\mathscr{L}\{u^{(2)}\} = {} & 3(A^2 e^{2i\theta_1} + A^{*2} e^{-2i\theta_1} + 2AA^* + \tfrac{1}{4}B^2 e^{2i\theta_2} + \tfrac{1}{4}B^{*2} e^{-2i\theta_2} \\ & + \tfrac{1}{2}BB^* + AB\, e^{i(\theta_1+\theta_2)} + AB^*\, e^{i(\theta_1-\theta_2)} + A^*B\, e^{i(\theta_2-\theta_1)} + A^*B^*\, e^{-i(\theta_1+\theta_2)}) \\ & + 2i\omega U_A A_{X_1} e^{i\theta_1} - 2i\omega U_A A^*_{X_1} e^{-i\theta_1} \\ & + 2i\omega U_B B_{X_1} e^{i\theta_2} - 2i\omega U_B B_{X_1} e^{-i\theta_2}.\end{aligned} \tag{3.7}$$

Now cancellation of terms on the right side of (3.7) that correspond to (nearly) free waves will lead to a bounded particular integral for $u^{(2)}$. Now $i\theta_1$ and $i\theta_2$ are themselves free wave phase functions; $2i\theta_1$ and $i(\theta_2-\theta_1)$ are nearly so, since

$$\begin{aligned} 2i\theta_1 &= i\theta_2 + 2ik_2c_2(U_A^{-1} - U_B^{-1})\,\delta x, \\ i(\theta_2-\theta_1) &= i\theta_1 - 2ik_2c_2(U_A^{-1} - U_B^{-1})\,\delta x. \end{aligned} \tag{3.8}$$

If, as suggested by (2.15), the detuning $\delta = O(\epsilon)$, then with

$$2k_2c_2(U_A^{-1} - U_B^{-1})\,\delta = N_2\epsilon,$$

the phase functions (3.8) become

$$\left.\begin{aligned} 2i\theta_1 &= i\theta_2 + iN_2X_1, \\ i(\theta_2-\theta_1) &= i\theta_1 - iN_2X_1, \end{aligned}\right\} \tag{3.9}$$

or the phases differ from free waves only on the long space scale. Clearly, then, the removal of the secular behaviour from (3.7) requires that

$$\left.\begin{aligned} U_A A_{X_1} &= iA^*Be^{-iN_2X_1}, \\ U_B B_{X_1} &= iA^2e^{+iN_2X_1}. \end{aligned}\right\} \tag{3.10}$$

These equations with N_2 set equal to zero are, of course, the amplitude equations determined previously by McGoldrick (1970*b*) for exact second harmonic resonance. We shall now investigate (3.10) and specify initial conditions at $X_1 = 0$ appropriate to the experiments of the next section.

Letting
$$W = iA^{*2}Be^{-iN_2X_1} \tag{3.11}$$

then (3.10) gives with some manipulation

$$\frac{d}{dX_1}(U_AAA^*) = W+W^* = -\frac{d}{dX_1}(U_BBB^*) = 2\,\mathrm{Re}\,(W), \tag{3.12}$$

which has the obvious integral

$$Z = U_A(\hat{A}^2 - AA^*) = U_B(BB^* - \hat{B}^2), \tag{3.13}$$

with
$$U_AAA^* + U_BBB^* = E(= U_A\hat{A}^2 + U_B\hat{B}^2). \tag{3.14}$$

Equation (3.14) is an 'energy-like' integral, E is a constant, and $\hat{A}$ and $\hat{B}$ are the moduli of the initial amplitudes of the two modes at $X_1 = 0$, say. Note that the energy integral is independent of the detuning N_2. Another, less obvious, independent integral can be constructed from $i(W-W^*)$. A little algebra shows that

$$i\frac{d}{dX_1}(W-W^*) = N_2(W+W^*) = -N_2\frac{dZ}{dX_1}, \tag{3.15}$$

using (3.12) and (3.13). Integrating (3.15), we get

$$-i(W-W^*) = N_2Z - 2L = 2\,\mathrm{Im}\,(W), \tag{3.16}$$

with L a constant of integration. Now since $2\,\mathrm{Re}\,(W) = -dZ/dX_1$,

$$WW^* = \mathrm{Re}^2(W) + \mathrm{Im}^2(W)$$

together with (3.16) easily yields

$$\left(\frac{dZ}{dX_1}\right)^2 = 4\left\{\left(\hat{A}^2 - \frac{Z}{U_A}\right)^2\left(\hat{B}^2 + \frac{Z}{U_B}\right) - (L - \tfrac{1}{2}N_2Z)^2\right\}, \tag{3.17}$$

which may be integrated explicitly in terms of elliptic functions,† and the ensuing details of the propagation depend crucially on the initial conditions $\hat{A}$ and $\hat{B}$ as well as the constant L, which can be shown to involve the initial relative phase difference between the interacting components.

† Integrals of the form of (3.14) and (3.16) and the subsequent reduction to quadrature were first found by Bretherton (1964) in a similar context.

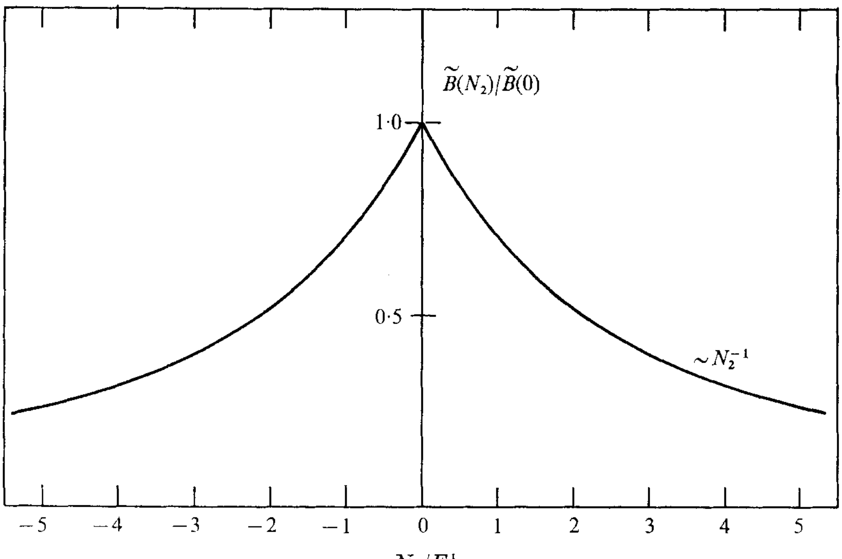

FIGURE 2. Second harmonic near-resonant response function, in terms of the detuning N_2.

For the initial conditions obtaining in the experiments, we choose $\hat{B} = 0$, corresponding to the generation of a pure fundamental mode at the wave maker. Then $E = U_A \hat{A}^2$, and with $U_A = 1$, $U_B = 2$, (3.17) becomes

$$\left(\frac{dZ}{dX_1}\right)^2 = 4\{\tfrac{1}{2}(E-Z)^2 Z - (L - \tfrac{1}{2}N_2 Z)^2\}. \tag{3.18}$$

Further, at $X_1 = 0$, the initial conditions together with (3.13) give $Z(0) = 0$, which in (3.18) implies $(dZ/dX_1)^2_{X_1=0} = -4L^2$, which in turn implies that, corresponding to this choice of initial conditions, the constant L must be zero in order for Z to be real. Then (3.18) becomes

$$(dZ/dX_1)^2 = Z\{2(E-Z)^2 - N_2^2 Z\} = \mathscr{G}_2(Z). \tag{3.19}$$

The nature of the propagation is clear from (3.19). The square of the modulus of the second harmonic, proportional to Z, oscillates periodically between the two roots of the cubic $\mathscr{G}_2(Z) = 0$ between which $G_2(Z) > 0$. One root is zero, of course, corresponding to $X_1 = 0$, and of the remaining two, the smaller denoted by $Z_{\max}$, is

$$Z_{\max} = E\left[1 - \frac{N_2^2}{4E}\left(\left(1 + \frac{8E}{N_2^2}\right)^{\frac{1}{2}} - 1\right)\right]. \tag{3.20}$$

(The remaining root is greater than E, which is physically unrealistic via (3.14).) $Z_{\max}$ is dependent on N_2, the amount of detuning, and the interpretation of the effects of detuning from resonance can best be seen in terms of a resonant response curve. If we call $\tilde{B}(N_2)$ the maximum amplitude attained by the second harmonic during the propagation ($\tilde{B} = (Z_{\max}/U_B)^{\frac{1}{2}}$), then figure 2 represents such a response curve. The ordinate represents the ratio of the maximum amplitude attained by the second harmonic off resonance to that which would be attained exactly on resonance, as a function of the amount of detuning, N_2. For large amounts of detuning, far from resonance, the response decreases as N_2^{-1}. Recalling that N_2 is proportional to ϵ times the departure of the frequency from resonance, we have demonstrated that there is a band of frequencies with bandwidth of order ϵ about the resonant frequency ω_2 for which the resonance mechanism is effective. The most noteworthy feature is that, according to figure 2, the maximum amount of energy transferred from the fundamental component into the second harmonic is decreased as the resonance is detuned. Finally, the response at resonance for these initial conditions is *not* periodic. For $N_2 = 0$, $Z = E$ is a double root of $\mathscr{G}_2(Z) = 0$, and (3.19) integrates directly in terms of hyperbolic functions, giving

$$Z(X_1) = E \tanh^2 (E^{\frac{1}{2}} X_1/\sqrt{2}). \tag{3.21}$$

The details of the exact resonance solution were the subject of an extensive experimental confirmation (McGoldrick 1970*a*), and need not be further considered here. Equation (3.21) represents the only non-periodic solution of (3.10) with the initial conditions used here; all near-resonant solutions are periodic. We shall contrast this fact with the conclusions that shall be drawn concerning higher harmonic resonances, to which we now turn.

3.2. *Third harmonic near resonance*

The problem of third harmonic resonance is somewhat more tedious than the preceding. The analysis must be carried to $O(\epsilon^2)$, and many of the algebraic details will not be inflicted on the reader. Third harmonic resonance will occur when the frequency of a fundamental mode satisfies $3\omega = f(3k)$, to which correspond $\omega_3 = \frac{4}{3}$ and $k_3 = (\frac{2}{3})^{\frac{1}{2}}$. For frequencies in the neighbourhood of ω_3, as suggested by (2.14) and (2.15), we shall choose for the $O(1)$ solution† of (3.2) the sum of the two oscillations

$$u^{(1)} = A\,e^{i\theta_1} + \tfrac{1}{3}B\,e^{i\theta_3} + [*], \tag{3.22}$$

in which the phase functions $\theta_1 = k(\omega)x - \omega t$ and $\theta_3 = k(3\omega)x - 3\omega t$ representing free waves, and the frequency ω is written as $\omega = \omega_3(1+\delta)$. Then the dispersion relation gives for the wavenumbers

$$\frac{k(3\omega)}{k(\omega)} = 3\left\{1 + c_3\delta\left(\frac{1}{U_B} - \frac{1}{U_A}\right)\right\} \tag{3.23}$$

to lowest order in δ, with c_3 and U_A the phase and group speeds evaluated at ω_3 and U_B the group speed evaluated at $3\omega_3$.

The $O(\epsilon)$ problem for $u^{(2)}$ yields two pieces of information. First, $A_{X_1} = B_{X_1} = 0$: that is, there is no variation of the amplitudes on the first long space scale. Then a particular integral of $\mathscr{L}\{u^{(2)}\} = 3u^{(1)2}$ is found to be

$$u^{(2)} = \{-\tfrac{9}{5}A^2 e^{2i\theta_1} + \tfrac{1}{315}B^2 e^{2i\theta_3} + \tfrac{6}{35}AB\,e^{i(\theta_1+\theta_3)} - \tfrac{6}{5}AB^*\,e^{i(\theta_1-\theta_3)}\} + \{*\} + 6AA^* + \tfrac{2}{3}BB^*, \tag{3.24}$$

no constituents of which are close to free waves, but bounded forced modes.

† We offer a small apology for the similarity of notation with the preceding second harmonic analysis. Rather than proliferate symbols, we have chosen a uniform notation, and hope that any confusion may not arise. Wherever that possibility may appear, we shall subscript: cf. θ_2 and θ_3.

Turning now to the $O(\epsilon^2)$ problem $\mathscr{L}\{u^{(3)}\}$, we shall examine the right-hand side term by term. The products $6u^{(1)}u^{(2)}$ and $\beta u^{(1)3}$ contain both free waves and waves close to free waves, as well as forced waves. The first and last groups of terms in parentheses vanish by virtue of the results of the $O(\epsilon)$ problem. The second group in parentheses contains free waves whose amplitudes are differentiated on the X_2-scale. As usual, in order that $u^{(3)}$ be bounded, the free waves must be eliminated from the inhomogeneous part of the equation. Omitting the straightforward details, the equations for the modulation of the amplitudes become

$$\left.\begin{aligned} U_A A_{X_2} &= i[c_{11}AA^* + c_{12}BB^*]A + id_3 A^{*2} B e^{-iN_3X_2}, \\ U_B B_{X_2} &= i[c_{21}AA^* + c_{22}BB^*]B + id_3 A^3 e^{+iN_3X_2}. \end{aligned}\right\} \qquad (3.25)$$

In arriving at the form of the complex exponentials in the last terms of (3.25), we have made use of the suggestion of (2.14) and (2.15): viz. the actual detuning δ is of order ϵ^2 for this problem. Explicitly, as in (3.8) and (3.9)

$$3k_3c_3(U_A^{-1} - U_B^{-1})\,\delta x \to N_3X_2$$

when $\delta = O(\epsilon^2)$, defining the detuning coefficient N_3. The real constants c_{ij} in (3.25) are

$$\left.\begin{aligned} &c_{11} = \tfrac{3}{8}(3\beta + \tfrac{126}{5}), \quad c_{22} = \tfrac{1}{12}(\beta + \tfrac{211}{35}), \\ &c_{21} = 3c_{12} = \tfrac{3}{4}(\beta + \tfrac{102}{35}), \end{aligned}\right\} \qquad (3.26)$$

and the interaction coefficient d_3 is

$$d_3 = \tfrac{3}{8}(\beta - \tfrac{54}{5}).$$

The $O(\epsilon^2)$ amplitude equations (3.25) possess two independent integrals with subsequent reduction to quadrature as in the previous case. If we let

$$W = iA^{*3}Be^{-iN_3X_2} \qquad (3.27)$$

then, with some manipulation,

$$\frac{d}{dX_2}(U_AAA^*) = d_3(W+W^*) = -\frac{d}{dX_2}(U_BBB^*) = 2d_3\,\mathrm{Re}\,(W), \qquad (3.28)$$

which has the integral

$$Z = U_A(\hat{A}^2 - AA^*) = U_B(BB^* - \hat{B}^2), \qquad (3.29)$$

with
$$U_AAA^* + U_BBB^* = E(= U_A\hat{A}^2 + U_B\hat{B}^2). \qquad (3.30)$$

Note that this integral is identical to that of the second harmonic resonance problem, and involves neither the constants c_{ij}, the interaction constant d_3, nor the detuning constant N_3. For the second integral, if we consider $\mathrm{Im}\,(W)$, then after a considerable amount of reduction, we get

$$\frac{d}{dX_2}(2\,\mathrm{Im}\,(W)) = \frac{1}{d_3}\left\{(N_3+\xi_1)\frac{dZ}{dX_2} + \xi_2\frac{dZ^2}{dX_2}\right\}, \qquad (3.31)$$

where the real constants ξ_1 and ξ_2 are

$$\left.\begin{aligned} \xi_1 &= \hat{A}^2\left(\frac{3c_{11}}{U_A} - \frac{c_{21}}{U_B}\right) + \hat{B}^2\left(\frac{3c_{12}}{U_A} - \frac{c_{22}}{U_B}\right), \\ \xi_2 &= \frac{1}{2}\left(\frac{c_{21}}{U_AU_B} - \frac{3c_{11}}{U_A^2}\right) + \frac{1}{2}\left(\frac{3c_{12}}{U_AU_B} - \frac{c_{22}}{U_B^2}\right). \end{aligned}\right\} \qquad (3.32)$$

Integrating (3.31), $$\mathrm{Im}(W) = \frac{1}{2d_3}\{(N_3+\xi_1)Z + \xi_2Z^2 + 2L\}, \qquad (3.33)$$

where the constant of integration L depends on the initial conditions. Then from (3.28), (3.29) and (3.33) we obtain a single differential equation for Z:

$$\left(\frac{dZ}{dX_2}\right)^2 = \frac{4d_3^2}{U_A^3 U_B}(U_A\hat{A}^2 - Z)^3 (U_B\hat{B}^2 + Z) - (2L + (N_3 + \xi_1) Z + \xi_2 Z^2)^2, \quad (3.34)$$

which again may be integrated in terms of elliptic functions.

In this case, a choice of initial conditions appropriate to the experiments simplifies (3.34) a bit. With $B(0) = 0$, then $U_A \hat{A}^2 = E$ and $W(0) = 0$, and $Z(0) = 0$, so from the integral (3.33) $L = 0$ corresponds to these initial conditions. Then (3.34) becomes, using $U_A = \frac{1}{3}U_B = (\frac{2}{3})^{\frac{1}{2}}$,

$$\left(\frac{dZ}{dX_2}\right)^2 = Z\{3d_3^2(E-Z)^3 - Z(N_3 + \xi_1 + \xi_2 Z)^2\} = \mathscr{G}_3(Z), \quad (3.35)$$

with $\xi_1 = \frac{1}{2}E(9c_{11} - 3c_{12})$ and $\xi_2 = \frac{1}{12}(18c_{12} - c_{22} - 27c_{11})$. The solutions of (3.35) are oscillatory, with Z oscillating periodically between the two smallest roots of the quartic $\mathscr{G}_3(Z) = 0$. One root is of course zero; the others may be obtained by solving the reduced cubic equation, which is more tedious than enlightening. Before investigating the upper bound of the amplitudes during the modulation, we shall investigate two special cases of particular importance.

The interaction coefficient d_3 depends on β, the as yet unspecified coefficient of the cubic nonlinearity in the original model problem (2.7). For the special case in which $\beta = \frac{54}{5}$, the interaction coefficient d_3 vanishes, and the amplitude equations are

$$\left.\begin{aligned} U_A A_{X_2} &= iA[c_{11}AA^* + c_{12}BB^*], \\ U_B B_{X_2} &= iB[c_{21}AA^* + c_{22}BB^*]. \end{aligned}\right\} \quad (3.36)$$

If we write the complex amplitudes in terms of their real amplitudes and real phases as

$$A = a e^{i\psi_A}, \quad B = b e^{i\psi_B}$$

then (3.26) become

$$\left.\begin{aligned} a_{X_2} = 0, \quad b_{X_2} &= 0, \\ U_A(\psi_A)_{X_2} &= c_{11}a^2 + c_{12}b^2, \\ U_B(\psi_B)_{X_2} &= c_{21}a^2 + c_{22}b^2, \end{aligned}\right\} \quad (3.37)$$

which have the immediate integrals for the amplitudes a = constant, b = constant (cf. (3.28) with $d_3 = 0$), and for the phases

$$\left.\begin{aligned} \psi_A &= U_A^{-1}(c_{11}a^2 + c_{12}b^2) X_2, \\ \psi_B &= U_B^{-1}(c_{21}a^2 + c_{22}b^2) X_2. \end{aligned}\right\} \quad (3.38)$$

If we return these to (3.22), then for this non-resonant special case

$$\begin{aligned} u^{(1)} &= A \exp i\{-\omega t + k(\omega)[1 + \epsilon^2(k(\omega) U_A)^{-1} (c_{11}a^2 + c_{12}b^2)]x\} \\ &\quad + \tfrac{1}{2}B \exp i\{-3\omega t + k(3\omega)[1 + \epsilon^2(k(3\omega) U_B)^{-1} (c_{21}a^2 + c_{22}b^2)]x\} + [*], \quad (3.39) \end{aligned}$$

with amplitudes A and B constant. The slowly varying phases ψ_A and ψ_B represent $O(\epsilon^2)$ Poincaré-type *wavenumber* shifts. For single gravity waves, these are of course the well-known Stokes corrections to the dispersion relation. That is, the coefficients c_{11} and c_{22} represent the self-interaction of waves to $O(\epsilon^2)$. The coefficients c_{21} and c_{12} on the other hand represent the lesser-known mutual

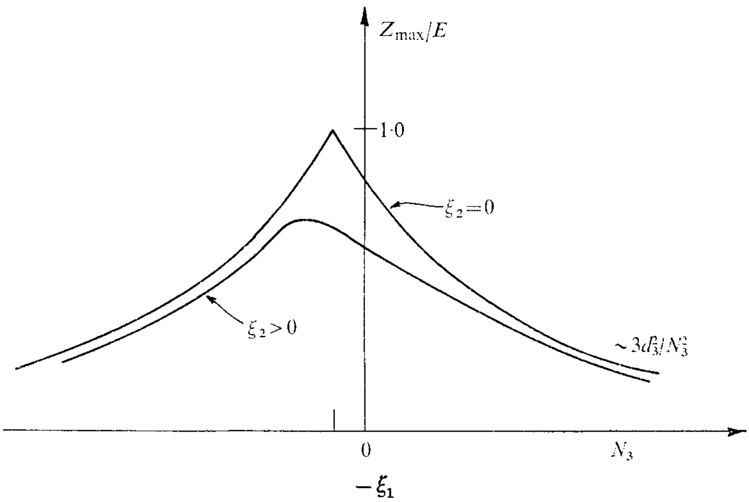

FIGURE 3. Typical third harmonic near-resonant response functions for several choices of the mutual and self-interaction coefficients.

(but non-resonant) finite amplitude corrections first found for water waves by Longuet-Higgins & Phillips (1962) by a less general method than that used here. All of these corrections arise from cubic interaction theory, and do not appear in the $O(\epsilon)$ problem considered earlier in this section.

For $d_3 \neq 0$, there is one more special case that should be investigated. Recalling that in the second harmonic problem the solution is monotonic for exact resonance, it is natural to seek similarity behaved solutions of (3.35). That is we wish to find a solution for which the energy of the fundamental mode can be transferred entirely and monotonically into the third harmonic, a situation representative of the maximum possible amount of energy transfer. This can be seen to be the case *only* if $Z = E$ is a multiple root of $\mathscr{G}_3(Z) = 0$. If E is a root at all, we must have $N_3 = -\xi_1 - \xi_2 E$. If E is a double root, then it is easy to obtain the remaining roots, which in ascending order are

$$Z = 0, \quad E\left(1+\frac{\xi_2^2}{3d_3^2}\right)^{-1}, \quad E, \quad E,$$

and the behaviour is still oscillatory. It is only for $\xi_2 = 0$ (in which case $Z = E$ is a triple root) *and* $N_3 = -\xi_1$ that the solution is monotonic. Simple algebra shows that in order for this to occur at all, the coefficient of the cubic nonlinearity in (2.7) must be

$$\beta = -20024/21705.$$

The solution of the amplitude equations for this case is

$$Z(X_2) = \tfrac{3}{4}d_3^2E^3X_2^2(1+\tfrac{3}{4}d_3^2E^2X_2^2)^{-1}. \tag{3.40}$$

It is remarkable that this 'maximum energy transfer' solution does not occur exactly at the frequency predicted by the resonance condition $3\omega = f(3k)$, but is in fact slightly detuned by amount $N_3 = -\xi_1$, which depends on the values of the self and mutual interaction coefficients c_{ij}. But this is not surprising, since these effects affect the dispersion relation itself to this order.

Again we may summarize the essential features of the interaction to this order in terms of a response function diagram, figure 3. We have plotted the maximum value attained by Z during the interaction as a function of the amount of detuning, N_3. The upper curve is drawn for the special case $\xi_2 = 0$, for which the solutions are periodic with the sole exception of the slightly detuned degenerate monotonic case, (3.40), which, of course, has the maximum response. For $\xi_2 \neq 0$, of which the lower curve is typical, all solutions are oscillatory, the relative maximum response must be less than that above, and occurs at a slightly detuned frequency. We shall see evidence of this influence of self and mutual interaction in the experiments described shortly.†

3.3. *Higher order near-resonances*

Higher order resonances may be analysed similarly. Even in the model problem the algebra is daunting. Nonetheless, we have learned enough of the essential features of these kinds of interactions to generalize a bit. Briefly considering fourth harmonic near-resonance, if we choose $u^{(1)} = A\,e^{i\theta_1} + \frac{1}{4}B\,e^{i\theta_4} + [*]$, then with the usual assumption $\omega = \omega_4(1+\delta)$ and usual choice of wavenumbers to satisfy the $O(1)$ problem, we get in turn:

$O(\epsilon)$: amplitudes not dependent on X_1, then particular integral for $u^{(2)}$.

$O(\epsilon^2)$: Poincaré wavenumber shifts of $O(\epsilon^2)$ from equations like (3.36) (non-resonant), and then a particular (bounded) integral for $u^{(3)}$.

$O(\epsilon^3)$ amplitude equations on the third long space scale X_3 of the form

$$\left.\begin{aligned} U_A A_{X_3} &= i\,d_4 A^{*3} B\,e^{-iN_4X_3} \\ U_B B_{X_3} &= i\,d_4 A^4 e^{+iN_4X_3}, \end{aligned}\right\} \tag{3.41}$$

where we have assumed $\delta = O(\epsilon^3)$ and

$$4k_4 c_4 (U_A^{-1} - U_B^{-1})\,\delta = N_4 \epsilon^3.$$

Then (3.41) are reduced in the usual way to

$$\left(\frac{dZ}{dX_3}\right)^2 = 4\sqrt{2}\,d_4^2 (E-Z)^4 Z - N_4^2 Z^2 = \mathscr{G}_4(Z), \tag{3.42}$$

having the usual oscillatory behaviour, the bounds being given by appropriate roots of the fifth-degree polynomial $\mathscr{G}_4(Z) = 0$, one of which is of course zero, corresponding to the initial conditions.

Continuing to fifth-harmonic near-resonance, with $u^{(1)} = A\,e^{i\theta_1} + \frac{1}{5}B\,e^{i\theta_5} + [*]$, then with $\delta = O(\epsilon^4)$, the sequence of problems to $O(\epsilon^2)$ is similar. To $O(\epsilon^3)$, we find that the $O(1)$ amplitudes are not dependent on X_3, whence a particular integral for $u^{(4)}$ leads at the next order, ϵ^4, to amplitude equations of the form

$$\left.\begin{aligned} U_A A_{X_4} &= iA[c_{111}A^2A^{*2} + c_{112}AA^*BB^* + c_{122}B^2B^{*2}] + i\,d_5 A^{*4} B\,e^{-iN_5X_4}, \\ U_B B_{X_4} &= iB[c_{211}A^2A^{*2} + c_{212}AA^*BB^* + c_{222}B^2B^{*2}] + i\,d_5 A^5 e^{+iN_5X_4}, \end{aligned}\right\} \tag{3.43}$$

† For gravity waves, certain triads of waves can produce a cubic resonance. The experiments of Longuet-Higgins & Smith (1966) and of McGoldrick, Phillips, Huang & Hodgson (1966) indicate quite clearly that the maximum energy transfer occurs slightly 'off-tune', and was accounted for explicitly by those authors.

with obvious reduction to $(dZ/dX_4)^2 = \mathscr{G}_5(Z)$, a sixth-degree polynomial in Z. The real constants c_{ijk} arise from quintic self and mutual interactions and are responsible for further Poincaré shifts of order ϵ^4.

The pattern that has emerged is clear. For nth harmonic resonance, there is a band of frequencies about ω_n with bandwidth $O(\epsilon^{n-1})$ for which the final resolution of the $O(1)$ wave field is given by solutions of $(dZ/dX_{n-1})^2 = \mathscr{G}_n(Z)$, an $(n+1)$th degree polynomial, with Poincaré detunings arising at each even order. The solutions are almost always periodic. The period of energy interchange, or interaction distance, is of order $k^{-1}(\omega)\,\epsilon^{1-n}$; that is, the energy transfer rate is progressively weaker as the order of the interaction increases. Finally, we state the following theorem: *In a conservative, weakly nonlinear one-dimensional, dispersive wave system, with dispersion relation $\omega = f(k)$ admitting a discrete sequence of frequencies which are solutions of $n\omega_n = f(nk)$ for $n = 2, 3, \ldots$, a single wave with frequency arbitrarily close to ω_n can* almost never *exist as a steady state, but must share its energy with a wave of frequency arbitrarily close to $n\omega_n$ which is capable of growth to the same order as the fundamental at the expense of the fundamental component itself.* That is, while this wave represents a theoretically possible state of equilibrium to lowest order, it is in fact dynamically unstable. The exceptions are those coincidental cases like (3.36) for which the interaction coefficient vanishes at some order. The result is capable of generalization to propagation in more than one direction, with the appropriate choice of a resonance condition. That, indeed, is what the resonant interaction theories of the last decade are all about.

4. Some experimental observations

In this section, we shall consider some (certainly not all) of the implications of the analysis of the last section in the neighbourhood of harmonic resonances. As pointed out in the second section, one-dimensional trains of capillary-gravity waves, which are weakly nonlinear, do have a dispersion relation with harmonic resonant solutions. The analogy is not exact, however, since the model equation provides no mechanism for dissipation whereas real water waves are well known to be weakly dissipative if the wave Reynolds number $R_w = \omega/\nu k^2$ is sufficiently large (ν is the kinematic viscosity). We shall not provide as detailed an accounting for the dissipative effects as has been done in our earlier (1970*a*) experiments on exact second harmonic resonance, but shall make adjustments of our interpretation of some of the observations presented here, based on the intuition gathered and strengthened in those earlier experiments.

The tank in which the experiments were performed is rectangular, with length 301 cm and width 62·7 cm filled with ordinary tap water to a depth of 41·0 cm, effectively placing the waves in infinitely deep water. Waves are created near one end by oscillating vertically a triangular shaped plunger which extends the width of the tank, and is about 5 cm high. The front face, inclined forward at an angle of about 20° with the vertical, is faced with a smooth thin sheet of glass which, if kept clean, prevents meniscus reversals from occurring, eliminating unwanted disturbances in the immediate vicinity of the wave maker.

The plunger is oscillated vertically by an electro-magnetic servo-mechanism consisting of the innards of a powerful high fidelity loud-speaker and a position-to-voltage transducer all driven by a purely sinusoidal electronic signal. A complicated comparison and feed-back circuit ensures that the motion of the plunger is a faithful replica of the driving signal, and has no spurious harmonics of the kind introduced by mechanical linkages; the amplitude and frequency stability is typically better than one part in 10^5. There is an absorbing beach at the far end of the tank.

Measurements of the wave field at any point in the tank are made with a capacitance-type wave detector having a remarkably linear response. With no more than the simplest attention to technique, we can routinely detect waves of amplitude 10^{-3} mm.†

In all of the experiments, the wave maker was driven with a purely sinusoidal motion with frequency in the appropriate neighbourhood of the selective resonances. The resulting wave form close to the plunger was as close as we could get to a monochromatic Fourier component. That is, the initial conditions are excellently approximated by $B(0) = 0$, which is appropriate to the analysis presented in the last section.

Returning now to dimensional variables, recall that the dispersion relation (2·5), $\omega^2 = gk + \gamma k^3$, has discrete resonant solutions given by

$$k_n = (g/n\gamma)^{\frac{1}{2}}, \quad \omega_n^2 = (g^3/\gamma)^{\frac{1}{2}}(n+1)/n^{\frac{3}{2}}. \tag{4.1}$$

The phase speed is given by

$$c_n^2 = (ng\gamma)^{\frac{1}{2}}[(n+1)/n] \tag{4.2}$$

for both harmonics, and the group speeds U_A and U_B for the components are

$$U_{A_n} = c_n(n+3)/2(n+1), \quad U_{B_n} = c_n(3n+1)/2(n+1). \tag{4.3}$$

For near-resonance we write $\omega = \omega_n(1+\delta)$, then with $\delta = O(\epsilon^{n-1})$ the detuning exponent in the nth order amplitude equations becomes (cf. (3.8) *et seq.*)

$$\begin{aligned} N_n\epsilon^{n-1} &= nk_n c_n(U_{A_n}^{-1} - U_{B_n}^{-1})\,\delta \\ &= 4\left(\frac{g}{n\gamma}\right)^{\frac{1}{2}}\frac{n(n^2-1)}{(3n+1)(n+3)}\,\delta, \end{aligned} \tag{4.4}$$

and the ratio of the $O(1)$ phase speed of the harmonic to that of the fundamental is, to lowest order,

$$c_B/c_A = 1 + \frac{4(n^2-1)}{(3n+1)(n+3)}\,\delta. \tag{4.5}$$

That is, if the frequency of the fundamental is slightly greater than the resonant frequency ω_n, then $\delta > 0$, and the phase speed of the harmonic is slightly greater than that of the fundamental,‡ and *vice versa*. We shall provide experimental evidence for this subsequently.

† We can in fact see (electronically) the evaporation from the free surface. This of course was compensated for by dribbling in fresh water.

‡ This is why (in the neighbourhood of resonance) the analysis cannot be reduced to a steady state by the usual artifice of transforming to a uniformly translating co-ordinate system. There is not even a steady lowest order basic state.

The sequence of resonant frequencies depends on the value of the surface tension coefficient γ. It is known that the effective value of this coefficient for a continuously deforming surface which, moreover, has adsorbed a surface film is not identical with the static values measured with a tensiometer. Further, the film greatly changes the damping characteristics of the waves, and the surface in time approaches closely the inextensible (or immobile) surface investigated by Lamb,† with the attendant more rapid wave damping for all but the shortest of capillary waves. For successively higher order resonances, the growth rate decreases by successive powers of the maximum wave slope $\epsilon = a\kappa$, and concurrently the wavenumber of the harmonic component increases, with inevitable increase of the dissipative effects. While for our second harmonic resonance experiments an 'equilibrium dirtiness' approach was tolerable, the competition between resonant growth and viscous dissipation requires for their distinction a relatively clean surface. No unduly elaborate precautions were felt worthwhile, but the surface film was flushed over a weir every 20 min or so. This periodic flushing reduced the dissipation slightly (measured by eye) and ensured that most of the experiments were performed on a somewhat fresh surface. The surface tension coefficient adopted was that appropriate to the measured temperature of the water, obtainable from standard reference works.‡

The first set of experiments were performed in the neighbourhood of third harmonic resonance. It is clear from the analysis that the tuning for maximum response is not known *a priori*, since the detuning effect of the self and mutual interactions, characterized by the constants c_{ij} of the last section, are not known for capillary-gravity waves. For a temperature of 22 °C, the period corresponding to ω_3 is $T_3 = 119{\cdot}25$ ms, and so a band of frequencies in the neighbourhood of T_3 was investigated to determine the response in the neighbourhood of resonance. Now as the frequency of the plunger is changed, the wavenumber of the fundamental is changed also. The small parameter ϵ of the theory is the maximum slope of the wave, taken to be that of the fundamental measured at (or close to) the wave maker, ak_3 say. In order that ϵ remain constant as the wave maker period is varied, the amplitude of the fundamental was monitored with a wave probe placed about 10 cm from the wave maker, and adjusted so that the slope was indeed a constant (this procedure is followed in all of the experiments). For the experimental results of figure 4, the constant slope was maintained at $ak_3 = 0{\cdot}059$, to which corresponds a fundamental amplitude of about 0·28 mm at a period of 119·25 ms (T_3). For a range of periods close to this, the amplitude of the third harmonic was measured with a second wave probe about 40 cm from the wave maker with the aid of a sharp constant bandwidth (1·0 Hz) electronic band-pass filter (Quan-Tech model 304-R). Figure 4 is the result of this experiment. The ordinate s_3 is the ratio of the slope of the third harmonic to the maximum slope of the fundamental, $s_3 = 3bk_3/ak_3$, which at resonance is an $O(1)$ quantity; the abscissa is the actual wave maker period in ms. The maximum steepness ratio

† See Lamb (1932, §351).

‡ See, for instance, *Handbook of Chemistry and Physics*, 46th edn. (1965-6). In particular we used the formula $\gamma = (75{\cdot}77 - 0{\cdot}152T)$ (cm³ s⁻²), 15° < T < 25°. Spot checks with a tensiometer indicate that this is within a few tenths of a (cm³ s⁻²) of the measured value.

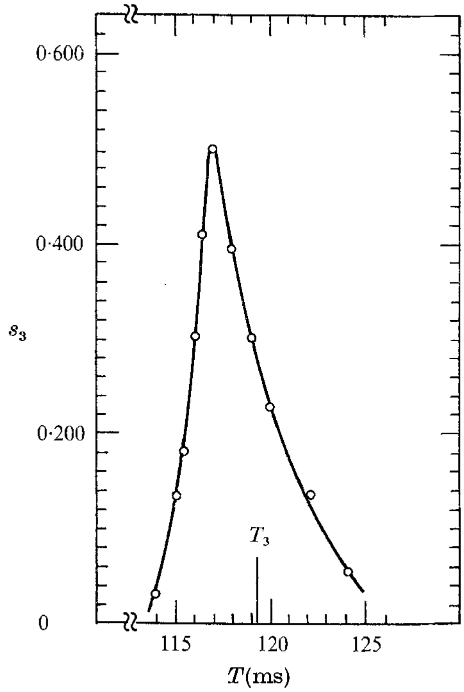

FIGURE 4. Measured amplitude response in the neighbourhood of third harmonic resonance. S_3 is the ratio of the steepness of the third harmonic measured 40 cm from the wave maker to the steepness of the fundamental near the wave maker.

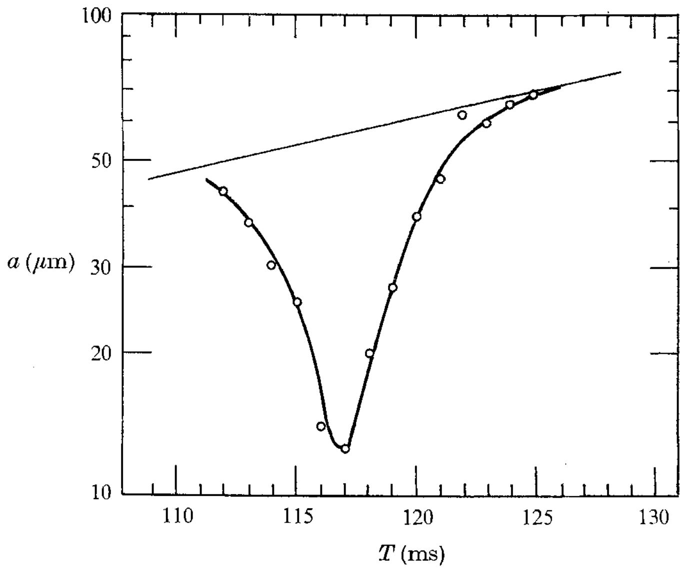

FIGURE 5. Measurements of the fundamental amplitude 120 cm from the wave maker, showing a cumulative effects of interaction and dissipation over the length of propagation.

observed is 0·5 and occurs at a period slightly different from that predicted by the $O(1)$ dispersion relation, as was anticipated. This measured response curve is not indicative of the absolute maximum response determined as in figure 3. since on the one hand we do not know in advance where (spatially) the third harmonic attains its maximum value, which depends of course on the amount of detuning.

On the other hand, however, for these experiments, the real viscous dissipation renders this comparison irrelevant. What is actually observed is the following sequence of events. As the almost pure fundamental wave leaves the wave maker, it loses its energy by two mechanisms: first, it begins to transfer energy into its third harmonic, which grows with distance; and second, it loses energy through dissipation. The viscous decrease of amplitude weakens the interaction, whence the growth rate of the third harmonic. The harmonic ripples themselves tend to be attenuated by the viscous dissipation, but to a greater degree than the fundamental. Ultimately, there is a point where the energy input to the ripples from the progressively weakening interaction is exactly balanced by their own viscous energy drain, after which both components decay and are ultimately extinguished. This same sequence of events was observed and quantified in our earlier experiments on second harmonic resonance (compare in particular figures 2 and 5 of that work), and is doubtless the situation here. What figure 4 does show clearly, though, is the relative strength of the interaction viewed in light of these two mechanisms in terms of the total growth of the resonant component over a *fixed* distance, and selectivity of the resonant tuning viewed in light of the degradation of the response for frequencies in the neighbourhood of the slightly shifted maximum response frequency, which occurs here at 117·0 ms period.

We may go further. We may measure the amplitude of the fundamental at a distance far in excess of the point where the interaction becomes dominated by dissipation. From this point onward the ripples will decay at an increasingly greater rate, being progressively deprived of their resonant source. Figure 5 shows a measurement of the fundamental component 120 cm from the source, as a function of the period of the wave maker. The third harmonic is virtually undetectable here; it has come and gone. The light sloping line in the figure represents the amplitude that the wave would have were there no nonlinear interaction, but solely viscous dissipation with logarithmic decrement $2\nu k^2/U_A$ cm^{-1}, where again it is the maximum slope near the plunger ϵ which is held constant. The actual measurements show a deficit from this value and the maximum deficit occurs, of course, where the interaction is the strongest, at $T = 117{\cdot}0$ ms. The difference between the light line and the measurements is a measure of the total amount of resonant energy transfer from the fundamental into the third harmonic, integrated over the path length of the propagation, which is as clear an indication of the strength of the resonance as is the constant distance response of figure 4.

Before turning to the next sequence of experiments, it is worthwhile to look at the spectral content of the wave form near resonance. Figure 6 shows such a determination at maximum response measured 40 cm from the source. The ordinate is the ratio of the maximum slope of the individual harmonics to that

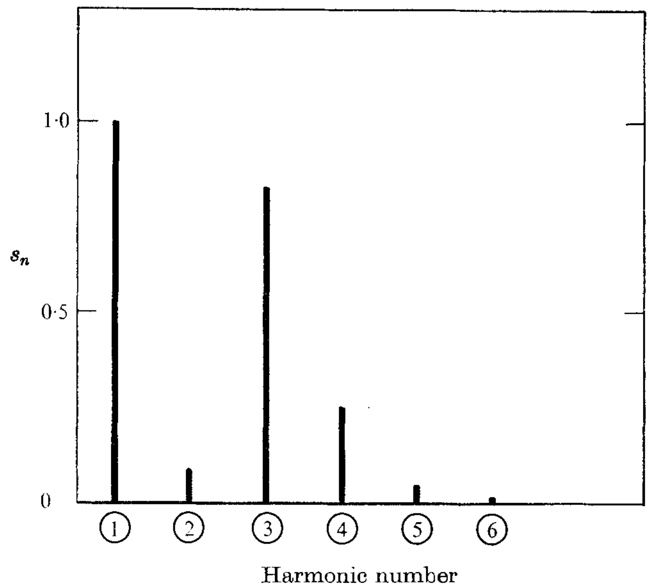

FIGURE 6. Amplitude spectrum at maximum third harmonic response. S_n is the ratio of the steepness of the nth harmonic to the *local* steepness of the fundamental.

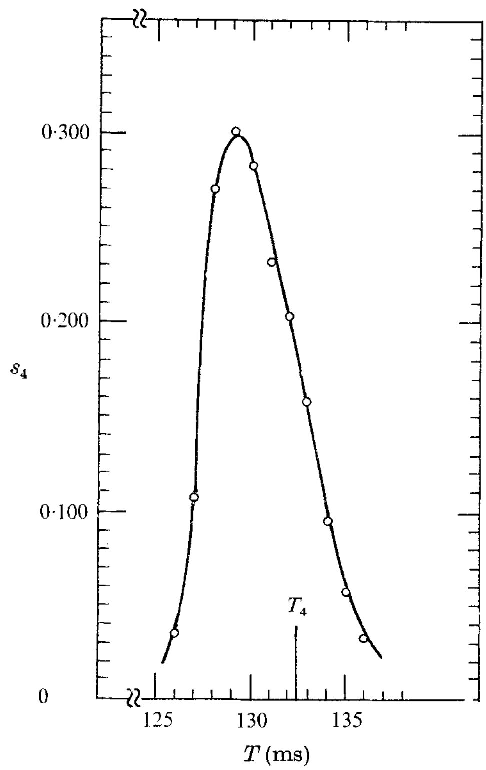

FIGURE 7. Measured amplitude response in the neighbourhood of fourth harmonic resonance.

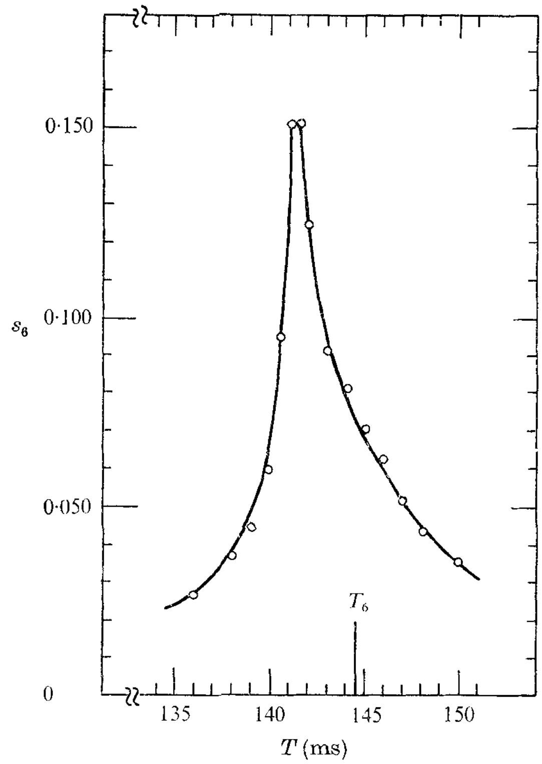

FIGURE 8. Measured amplitude response in the neighbourhood of sixth harmonic resonance.

of the fundamental measured now at the same point. The slope of the third harmonic is 83 % that of the local fundamental; the other components are considerably smaller.

Turning briefly to fourth harmonic resonance, at a water temperature of 20·8 °C, the solution of the resonance condition gives, corresponding to ω_4, a period $T_4 = 132{\cdot}43$ ms. Again, in the neighbourhood of this frequency, a constant-ϵ response curve has been measured with a probe again at 40 cm. The fundamental amplitude at T_4 near the plunger was 0·5 mm, to which corresponds a maximum slope $ak_4 = 0{\cdot}092$, slightly larger than for the previous experiment. Figure 7 is the measured response curve. The peak response occurs slightly off the predicted $O(1)$ resonance, of course, and the maximum response S_4 is about 0·30, which is somewhat less than for third harmonic resonance. But this is expected since this interaction is weaker than the preceding while the measurements were made at the same location. Integrated decay measurements and spectral content (with fourth harmonic predominant) are similar to figures 5 and 6 and need not be presented here.

Continuing the experiments to sixth harmonic resonance, we have for this case ak_6 a maximum of 0·15 for which the fundamental measured at the wave maker had amplitude 0·98 mm. Figure 8 is the measured response, again at 40 cm from the source, with no further dissimilarities from the other responses. The maximum response is naturally smaller than before, which is expected. Nonetheless, in spite of the fact that the nonlinear terms responsible for this interaction are of order ϵ^5, with corresponding reduction in energy transfer rates, the sixth harmonic emerges at resonance as an $O(1)$ quantity.

We shall present just one further experimental observation which was promised at the beginning of this section. It is explicit from (4.5) that the phase speeds of the two harmonic components are not identical to lowest order in δ except for $\delta = 0$, corresponding to the satisfaction of the $O(1)$ resonance condition. A perhaps unnecessarily elaborate, but nonetheless convincingly satisfying, graphical demonstration of this fact can be shown. Figure 9 (plate 2) is a single photograph consisting of seven separate traces on an oscilloscope. The sweeping rate of the oscilloscope has been adjusted to display almost exactly two periods of wave forms. The period of the wave maker is longer than that required for sixth harmonic resonance (T_6) by 4·0 ms. That is, the frequency is less than ω_6, and the detuning is characterized by $\delta \doteq -0{\cdot}018$. The wave forms displayed are synthesized in the following way. The actual signal from the wave probe is passed separately through two sharp band-pass filters, the first of which extracts solely the fundamental frequency and the other, solely the sixth harmonic. These signals are added together after suitable rescaling to produce a pleasing rippled pattern. The topmost trace of the picture, which is initiated by a zero-crossing of the filtered fundamental, is written on the oscilloscope which stores the traces as a bright line (for several hours). Then the wave probe is moved 1·0 cm farther away from the source, the d.c. level of the oscilloscope trace is moved downward by a known amount, and the second trace is then written on the oscilloscope, again triggered by the *same phase point* of the filtered fundamental, viz. the zero-crossing. The remaining five traces are written by the same procedure: physical displacement of the probe by 1·0 cm followed by level displacement and constant phase point triggering. What is displayed, then, in this photograph is the phase of the sixth harmonic wave *relative* to the fundamental as a function of distance in the direction of propagation. If the harmonic has a phase speed less than the fundamental, the ripple pattern should be delayed relatively with increasing distance: that is, in the frame of reference of the photograph, the rippling should appear to propagate to the right. That it indeed does is quite clear from the photograph, particularly when viewed at glancing incidence from below. It is a simple matter to determine the relative phase speed. For this determination the ratio of the speed of the harmonic to that of the fundamental is

$$c_B/c_A = 0{\cdot}96.$$

From (4.5), $$c_B/c_A = 1 + \tfrac{140}{171}\delta = 0{\cdot}98.$$

The agreement is noteworthy.

There are direct ways to measure the individual phase speeds separately, by a Lissajous figure method. The results, while inherently more accurate, are less demonstrative of the conclusion. We shall not present here similar results which do show, for positive δ, that the ratio (4.5) is in fact greater than unity.

We have not presented any detailed measurements concerning the spatial growth and decay of the interacting components. While we do have many measurements of such quantities near and far from resonance, we shall defer them in favour of a description of the process based on visual observation. The rippling is most easily observed for the higher order resonances because of the difference in length scales of the interacting components. In all cases, however, the rippling can be seen by eye. Not too close to resonance, the rippling is relatively weak and its maximum occurrence is relatively far from the source. As the frequency of the wave maker is adjusted closer and closer to resonance, the rippling becomes more and more pronounced and the location of the visual maximum of the ripples moves closer to the source, being closest of course when the frequency corresponds to one of the maxima of the response diagrams presented earlier. This is indicative of the selectivity of the interactions, and in agreement with the analysis of the model problem. Again, if we maintain the source at a constant frequency and vary the amplitude of the fundamental by increasing the stroke, we observe also both that the maximum rippling increases and that the location of this maximum moves towards the source. This too is in agreement with our ideas concerning the competition between the effects of resonant ripple growth (which increases with amplitude of the fundamental at its expense) and effects of dissipation.

The shallow water rippling experiments of Kim & Hanratty (1971) deserve further comment here, since their conclusions about the generating mechanism are different from those of the present paper. For third and fourth harmonic generation, we propose cubic and quartic nonlinear interactions respectively, while they explain the harmonic distortion with quadratic interactions (i.e. $O(\epsilon)$ in our terminology).

For the $O(1)$ wave field, Kim & Hanratty write

$$\zeta = \sum_{\alpha=0}^{N} A_\alpha e^{i\alpha\theta} + [*] \quad \text{(K \& H 2)}$$

with a corresponding expression for the velocity potential, and make no *a priori* assumptions about the magnitudes of the amplitudes A_α. For the remainder of their paper, they concentrate on the first four harmonic modes ($N = 4$). For infinitely deep water it is clear from our dispersion relation (2.5) that not all four modes can satisfy the $O(1)$ (i.e. linear) equations simultaneously, but in fact at most two of the modes can be $O(1)$ depending on which resonance is being investigated; the remaining two amplitudes must be at most $O(\epsilon)$, and are to be determined as particular integrals of the $O(\epsilon)$ dynamic equations once the secularity has been removed.

For shallow water, the situation should be different. The dispersion relation (in dimensional terms) is

$$\omega = \{(gk + \gamma k^3) \tanh kH\}^{\frac{1}{2}},$$

which involves a ratio of two length scales. If L is a wavelength scale ($= 2\pi/k$) and we call $\Delta = H/L$, then the dispersion relation for small Δ is

$$\omega = (gH)^{\frac{1}{2}} k \left\{1 - \frac{4\pi^2}{3}\left(1 - \frac{3\gamma}{gH^2}\right)\Delta^2 + O(\Delta^4)\right\}^{\frac{1}{2}}. \tag{4.6}$$

Now our ϵ is the ratio of an amplitude to a wavelength, and the dynamical problem posed in § 2 involves two independent small parameters. If $\Delta \ll 1$ then the waves in addition to being weakly nonlinear ($\epsilon \ll 1$) are *weakly dispersive.* It is only for small Δ that a wave field of the form (K & H 2) can satisfy the $O(1)$ problem. Further, if the fluid depth is close to $(3\gamma/g)^{\frac{1}{2}}$ (about 0·48 cm for water), then writing $H^2 = 3\gamma(1+\Delta_1)/g$, (4.6) becomes, for small Δ_1,

$$\omega = (gH)^{\frac{1}{2}} k \left\{1 - \frac{4\pi^2}{3}\Delta_1 \Delta^2 + O(\Delta^4)\right\}^{\frac{1}{2}}. \tag{4.7}$$

As $\Delta_1 \rightarrow 0$, the waves are even more weakly dispersive, since the correction to the dispersion relation is at most $O(\Delta^4)$, and the $O(1)$ wave field used by Kim & Hanratty would be even more appropriate: they do comment on the weakness of the dispersion in this particular limit.

Mei & Ünlüata (1972) have performed similar experiments on harmonic generation on a much larger physical scale, where the fluid depth is much larger than $(3\gamma/g)^{\frac{1}{2}}$. The effects of surface tension are negligible. In their analysis they choose the scaling $\epsilon\Delta^{-2} = O(1)$,† which balances the amplitude dispersion effects with those of frequency dispersion (the scaling is that for solitary waves). For the dynamical equations, they choose the more appropriate Boussinesq water wave equations dictated by the choice of scaling. With this model they show that second and third harmonic generation can be produced by a quadratic $O(\epsilon)$ interaction and provide experimental verification of their analytical results.

It is quite clear from the work of Mei & Ünlüata (and its possible extension to higher harmonic generation) that the harmonics higher than the second can indeed be produced by a quadratic interaction: it is necessary that the waves be weakly dispersive in the sense described above. The essential difference between the work of the above pairs of authors and that described in this paper is that our model (and capillary-gravity waves on deep water) are fully dispersive with the consequence that resonant generation is a result of nth order interactions, as we have shown in the preceding section.

5. Further comments and summary

While it is true that the analysis of § 3 can formally be carried to any desired order, there are several reasons to doubt the applicability of such a procedure to the case of real water waves, at least as n becomes sufficiently large. First, the inevitable dissipative effects become increasingly more important as the wavelength of the resonant ripples diminishes. These effects have been treated only heuristically here, for obvious simplicity. More important, however, is the fact that, with increasing n, the ratio of the wavelength of the ripples to that of the fundamental responsible for their production becomes increasingly small according to $k(\omega)/k(n\omega) = \Lambda = 1/n$. In this resonance calculation, as well as in many others, the tacit assumption is ordinarily made that the wavelengths (whence frequencies) are all of the same order, with the result that the weakly

† Their ϵ is amplitude/depth.

nonlinear problem depends solely on a *single* small parameter, ϵ, say, which physically is a measure of the nonlinearity produced by finite amplitude effects. But for large n, Λ itself is another small parameter, and the actual problem depends ultimately on two small parameters, ϵ and Λ. In terms of a parametric (ϵ, Λ) space, we have solved the problem for the asymptotic limit $\epsilon/\Lambda = o(1)$ as $\epsilon \to 0$.

The rippling problems analysed by Longuet-Higgins (1963) and later by Crapper (1970) depend crucially on the assumption that Λ is small, so that the ripples can be treated as a perturbation on some given basic state, which they take to be the steady gravity wave itself. That is, if a typical horizontal length scale of the ripples is characterized by their wavelength l, and a horizontal length scale of the gravity wave is taken to be its wavelength L, then for l/L small (our Λ), the amplitude of the ripples, characterized by a, is taken to be dependent on a long space scale X given by $X = xl/L = \Lambda x$, or $a = a(X)$. The amplitude of the gravity wave A is taken to be independent of any horizontal length scale (viz. A = constant), but the phase function belonging to the gravity wave is assumed *ab initio* to be a function of $\Lambda x = X$. This horizontal spatial scaling is purely kinematical. Their approximation of the dynamical equations implies that a relative measure of the dynamical nonlinearities is, as usual, $\epsilon = A/L$. Then under the supposition that ϵ and Λ are of the same order they carry on. Their results show that for this choice of parameter ranges, the amplitude and phase of the ripples, both slowly varying over the scale of the long wave, are such that the ripples are confined for the most part to the forward face of the wave (ahead of the crest in the propagation direction), and the amplitude of the ripples decreases with distance from the crest; the wavelength of the ripples also decreases with distance from the crest. The magnitude of the amplitude and wavelength variation is shown to be proportional to the steepness of the gravity wave. Finally, they point out that their analyses (in which $\epsilon/\Lambda = O(1)$) breaks down as the wavelength of the gravity wave diminishes, i.e. as Λ becomes large. But this is precisely where our present analysis is valid, and our analysis becomes invalid when, for fixed $\epsilon, \Lambda \to \epsilon$. In this sense, the respective analyses are complimentary and not competitive. It is no wonder that the observations mentioned in § 1 were not in agreement with the earlier theories!

The experimental observations of Cox (1958) mentioned previously deserve further comment. It is clear now that they belong to the parameter range of the present paper. Cox performed his investigation in a combination wave tank/wind tunnel, and made note of the fact that waves created by a wave maker alone operating at about 6·6 c/s *without* the wind blowing created a good deal of rippling. With the addition of a wind blowing in the direction of the waves, the rippling was augmented. This now is easy to explain. The windless ripples were created by the resonance mechanism. The action of the wind is to feed energy into the longer waves maintaining them against the drain from both resonant transfer and viscous dissipation, and energy input manifests itself in the augmentation of the ripples with concurrent extension of their region of occurrence. It is conceivable that there is a wind speed for which the energy input to the long waves is precisely balanced by the resonant and viscous drain, which implies that the long wave serves as a catalyst for the indirect generation of ripples by the wind. Moreover, the effect of the rippling for any wind speed is to inhibit the onset of breaking of the long waves. This conclusion has already been drawn by Longuet-Higgins and by Crapper, but we now have a method to quantify it further, since the interaction coefficients are (in principle) calculable. On the other hand, the algebraic complexities are so severe that we suggest that the phenomenon be should investigated in more experimental detail.

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FIGURE 1. Fundamental wave with harmonic ripples. The wave maker frequency is 6·6 c/s. Note that the rippling is not confined to the front face, but appears all over the longer wave. Propagation is from right to left.

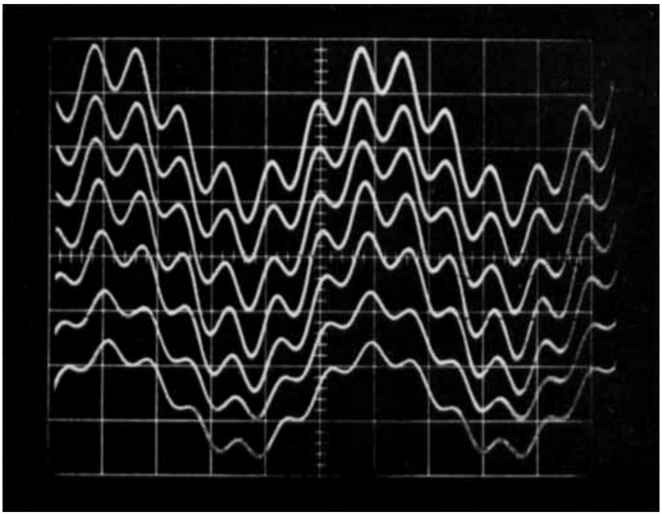

FIGURE 9. Reconstructed wave form near sixth harmonic resonance for measuring the relative phase speed of the interacting components.